TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 256, December 1979

SUBNORMAL OPERATORS QUASISIMILAR TO AN ISOMETRY[1]

BY

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ABSTRACT. Let $V = V_0 \oplus V_1$ be an isometry, where V_0 is unitary and V_1 is a unilateral shift of finite multiplicity n. Let $S = S_0 \oplus S_1$ be a subnormal operator where $S_0 \oplus S_1$ is the normal decomposition of S into a normal operator S_0 and a completely nonnormal operator S_1. It is shown that S is quasisimilar to V if and only if S_0 is unitarily equivalent to V_0 and S_1 is quasisimilar to V_1. To prove this, a standard representation is developed for n-cyclic subnormal operators. Using this representation, the class of subnormal operators which are quasisimilar to V_1 is completely characterized.

0. Introduction. The purpose of this paper is to study quasisimilarity within the class of subnormal operators. Two Hilbert space operators A and B are quasisimilar if there exist operators X and Y which are one-to-one, have dense range, and satisfy $XA = BX$ and $AY = YB$. Quasisimilarity was introduced by Sz.-Nagy and Foiaş, who gave a simple characterization of the class of operators which are quasisimilar to a unitary operator [13]. They also showed that if A is quasisimilar to a unitary operator W, then there is a one-to-one correspondence between the hyperinvariant subspaces of A and the hyperinvariant subspaces of W. In general, it is known that if A and B are quasisimilar and A has a hyperinvariant subspace, then so does B [7].

An operator S is subnormal if S has an extension T to a larger Hilbert space on which T is normal. Let μ be a positive, finite Borel measure with compact support in $\mathbf{C}$ and let U_μ be the operation of multiplication by z on $H^2(\mu)$, the closure in $L^2(\mu)$ of the polynomials in z. Up to unitary equivalence, the operator U_μ is the most general cyclic subnormal operator. If m is Lebesgue measure on $\mathfrak{T} = \{z: |z| = 1\}$, then $H^2 = H^2(m)$ is the classical Hardy space and U_m is the unilateral shift. A measure μ is of type $\mathfrak{S}$ if

(1) μ is carried by $\{z: |z| \leq 1\}$,

(2) the restriction of μ to $\mathfrak{T}$ is absolutely continuous,

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Received by the editors April 22, 1976 and, in revised form, December 21, 1976 and October 4, 1978.

AMS (MOS) subject classifications (1970). Primary 47B20.

Key words and phrases. Subnormal operator, isometry, quasisimilar, normal decomposition, unilateral shift, dominant operator.

[1]The paper is part of the author's dissertation submitted in partial fulfillment of the requirements of the Ph. D. at The University of Michigan. The author gratefully acknowledges his indebtedness to Professor Allen Shields.

(3) $\int \log(d\mu/dm)\, dm > -\infty$.

Recently, S. Clary proved that $U_\mu \sim\sim U_m$ if, and only if, μ is of type $\mathcal{S}$ [3], [4]. In this paper we will extend his result by replacing U_m by more general isometric operators.

Each Hilbert space operator A has an orthogonal decomposition $A = A_0 \oplus A_1$, where A_0 is normal and A_1 is completely nonnormal. By completely nonnormal we mean that the restriction of A_1 to a reducing subspace is never normal. Let S and S' be subnormal operators with decompositions $S_0 \oplus S_1$ and $S_0' \oplus S_1'$, respectively, and suppose S and S' are quasisimilar. We will show that S_0 and S_0' must be unitarily equivalent. It is not necessary, however, that S_1 and S_1' be quasisimilar even if S' is an isometry. On the other hand, if S_1' is the unilateral shift of finite multiplicity n, then we will conclude that S_1 and S_1' are quasisimilar.

This brings us to the question of which subnormal operators are quasisimilar to the shift of multiplicity n. Such an operator must be n-cyclic. An operator A on a Hilbert space H is n-cyclic, or cyclic of finite order n, if there is a set of n vectors $\{x_1, \dots, x_n\} \subset H$ such that the smallest subspace of H invariant under A and containing $\{x_1, \dots, x_n\}$ is all of H, while no set of $n-1$ vectors has this property. An n-cyclic subnormal operator has an n'-cyclic normal extension where $n' \leq n$. Using this fact, a standard representation is developed for n-cyclic subnormal operators. This model allows us to characterize the class of subnormal operators which are quasisimilar to the shift of multiplicity n.

1. Notation and definitions. All operators considered here are bounded Hilbert space operators. An operator X is quasiinvertible if X has dense range and no kernel. If two operators A and B are quasisimilar, we will write $A \sim\sim B$ or $A \sim\sim B$ via (X, Y), where X and Y are quasiinvertible operators such that $XA = BX$ and $AY = YB$. If A and B are unitarily equivalent, then we will write $A \cong B$ or $A \cong B$ via X, where X is unitary and $XA = BX$.

We will implicitly use the following facts concerning quasisimilarity throughout this paper.

LEMMA. *Suppose $A: H \to H$, $B: H' \to H'$ and $X: H \to H'$ are operators such that X has dense range and $XA = BX$. If A is n-cyclic, then B is cyclic of order at most n.* $\square$

COROLLARY. *If $A \sim\sim B$ and A is n-cyclic, then B is also n-cyclic.*

Let $\mathcal{S}$ be a subset of H. The smallest invariant subspace of $A: H \to H$ which contains $\mathcal{S}$ is denoted $\mathfrak{M}(\mathcal{S}; A)$. Similarly, the smallest subspace of H which contains $\mathcal{S}$ and reduces A is denoted $\mathfrak{N}(\mathcal{S}; A)$. If $\mathcal{S} = \{x_1, \dots, x_n\}$ is finite, then we write $\mathfrak{M}(x_1, \dots, x_n; A)$ and $\mathfrak{N}(x_1, \dots, x_n; A)$ for $\mathfrak{M}(\mathcal{S}, A)$ and $\mathfrak{N}(\mathcal{S}, A)$, respectively.

By *measure* we will always mean a positive, finite measure on the Borel subsets of $\mathbf{C}$. We will say that a measure μ is carried by a Borel set E if $\mu(\mathbf{C} \setminus E) = 0$. The support of a measure μ, denoted supp μ, is the complement of the union of all open sets of μ-measure zero. The measure μ restricted to a Borel set E is the measure $\mu|_E$ defined by $\mu|_E(F) = \mu(E \cap F)$. We will be particularly interested in measures which are carried by $\bar{D}$, where $D = \{z: |z| < 1\}$. If μ is such a measure, then we will write μ_0 for $\mu|_{\mathfrak{T}}$. (Recall that $\mathfrak{T}$ is the boundary of D.)

Let μ be a measure with compact support. Let W_μ: $L^2(\mu) \to L^2(\mu)$ be the operation of "multiplication by z". Thus $(W_\mu f)(z) = zf(z)$ for each $f \in L^2(\mu)$. Up to unitary equivalence the operator W_μ is the most general cyclic normal operator. For f and g in $L^2(\mu)$ we will write $(f, g)_\mu$ for their inner product and $\|f\|_\mu$ for the $L^2(\mu)$-norm of f.

2. Normal operators. We collect here some properties of normal operators which we will use in later sections. The above representation for cyclic normal operators generalizes to the noncyclic case as follows.

Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be a finite sequence of measures. Let $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$ be measurable functions mapping $\mathbf{C}$ into $\mathbf{C}^n$. We will say that $F = G$ a.e. $[\boldsymbol{\mu}]$ if $f_j = g_j$ a.e. $[\mu_j]$, $1 < j < n$. This defines an equivalence relation on the set of such functions, and, following the usual custom, we will ignore the distinction between a function and the equivalence class to which it belongs. Let $L^2(\boldsymbol{\mu})$ be the set of (equivalence classes of) measurable functions F: $\mathbf{C} \to \mathbf{C}^n$ such that

$$\left\{ \sum_{k=1}^{n} \int |f_k|^2 \, d\mu_k \right\}^{1/2} < \infty, \tag{1}$$

where $F = (f_1, \ldots, f_n)$. Using the left-hand side of (1) as norm, $L^2(\boldsymbol{\mu})$ becomes a separable Hilbert space. Define an operator $W_{\boldsymbol{\mu}}$: $L^2(\boldsymbol{\mu}) \to L^2(\boldsymbol{\mu})$ by $(W_{\boldsymbol{\mu}}F)(z) = zF(z) = (zf_1(z), \ldots, zf_n(z))$.

These definitions may also be made for an infinite sequence of measures $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$. In this case we consider functions $F = (f_1, f_2, \ldots)$ mapping $\mathbf{C}$ into the space of sequences of complex numbers and assume each f_j is measurable.

LEMMA 2.1. *A normal operator T: $K \to K$ is n-cyclic if, and only if, n is the smallest integer for which there exist n vectors $x_1, \ldots, x_n \in K$ such that $K = \mathfrak{M}(x_1, \ldots, x_n; T)$.*

PROOF. Suppose T is n-cyclic and suppose $K = \mathfrak{M}(x_1, \ldots, x_j; T)$. Let $\tilde{x}_1 = x_1$, and let $\tilde{x}_i$ be the orthogonal projection of x_i into

$$K \ominus \mathfrak{M}(x_1, \ldots, x_{i-1}; T), \qquad 2 < i < j.$$

Choose y_i such that $\mathfrak{M}(y_i; T) = \mathfrak{N}(\tilde{x}_i; T)$ (cf. [1, Theorem 6]). We have

$$\mathfrak{M}(y_1, \ldots, y_j; T) \supset \mathfrak{N}(\tilde{x}_1; T) \oplus \cdots \oplus \mathfrak{N}(\tilde{x}_j; T) = K.$$

Therefore, $n \leq j$. $\square$

LEMMA 2.2. *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a sequence of measures. The following statements are equivalent*:

(1) *W_μ is n-cyclic*;

(2) *there exists a nonzero measure ν such that $\nu \ll \mu_j$, $1 \leq j \leq n$*;

(3) *there exists a Borel set E such that $\mu_j(E) > 0$ and $\mu_j|_E \ll \mu_k$, $1 \leq j$, $k \leq n$.*

PROOF. Suppose W_μ is n-cyclic. For each j, $1 \leq j \leq n-1$, we may write $\mu_n = \nu_{ja} + \nu_{js}$, where $\nu_{ja} \ll \mu_j$ and $\nu_{js} \perp \mu_j$. Let E_j be a Borel set such that $\nu_{js}(\mathbf{C} \setminus E_j) = \mu_j(E_j) = 0$, and set $E_j' = \operatorname{supp} \mu_j \setminus E_j$. Let

$$F_j = \left(0, \ldots, 0, \chi_{E_j'}, 0, \ldots, 0, \chi_{E_j}\right), \qquad 1 \leq j \leq n-1.$$

Both $(0, \ldots, 0, 1, 0, \ldots, 0)$ and $(0, \ldots, 0, \chi_{E_j})$ are in $\mathfrak{N}(F_j; W_\mu)$. Let

$$E = \bigcup_{j=1}^{n-1} E_j.$$

It follows that $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1, 0)$ and $(0, \ldots, 0, \chi_E)$ are all elements of $\mathfrak{N}(F_1, \ldots, F_{n-1}; W_\mu)$. Since $\mathfrak{N}(F_1, \ldots, F_{n-1}; W_\mu) \neq L^2(\mu)$, we must conclude that $\mu_n(\mathbf{C} \setminus E) > 0$. By construction, $\nu_{js}(\mathbf{C} \setminus E) = 0$, $1 \leq j \leq n-1$. Therefore, the measure $\mu_n|_{\mathbf{C} \setminus E}$ satisfies statement (2).

Now suppose the measure ν satisfies (2). Let E be a Borel set such that ν is carried by E and ν and $\mu_j|_E$ are mutually absolutely continuous. Then

$$\mu_j|_E \ll \nu \ll \mu_k|_E \ll \mu_k,$$

and $\mu_j(E) > 0$ because $\nu(E) > 0$.

Finally, suppose there is a Borel set E satisfying statement (3). Let $\nu = \mu_1|_E$ and note that $\mu_j|_E$ and ν are mutually absolutely continuous. Suppose $L^2(\mu) = \mathfrak{N}(F_1, \ldots, F_{n'}; W_\mu)$. As shown in the proof of Lemma 2.1, we may assume that

$$\mathfrak{N}(F_j; W_\mu) \perp \mathfrak{N}(F_k; W_\mu), \qquad j \neq k.$$

Because of this orthogonality condition we may conclude that there are measurable functions g_{jk} such that

$$g_{j1}F_1 + \cdots + g_{j,n'}F_{n'} = (0, \ldots, 0, 1, 0, \ldots, 0), \qquad 1 \leq j \leq n \qquad (2)$$

(the "1" appearing in the jth position). Let $A(z)$ be the matrix $(g_{ij}(z))$. Then the equations (2) imply that $A(z)$, considered as a linear transformation from $\mathbf{C}^{n'}$ into $\mathbf{C}^n$, is surjective for almost every (with respect to ν) z. This is possible only if $n' \geq n$. Therefore W_μ is n-cyclic by Lemma 2.1.

THEOREM 2.3. *Let T: $K \to K$ be a normal operator on a separable Hilbert space K. Then there exists a sequence of measures $\mu = (\mu_1, \mu_2, \dots)$ such that $\mu_1 \gg \mu_2 \gg \dots$ and $T \cong W_\mu$. If T is cyclic of finite order n, then we may take $\mu = (\mu_1, \dots, \mu_n)$.*

PROOF. See [2, Theorem 10.3]. The last statement of the theorem follows from Lemma 2.2. □

Let T: $K \to K$ be normal. One result of the spectral theorem is that for each $x \in K$ there exists a unique measure ν_x such that

$$(T^j x, T^k x) = \int z^j \bar{z}^k \, d\nu_x(z).$$

In addition, there is a unitary operator V: $K \to L^2(\nu_x)$ such that $Vx = 1$ and

$$T|_{\mathfrak{M}(x;T)} \cong W_{\nu_x} \quad \text{via } V.$$

If K is separable, the measure ν_x may be found as follows.

LEMMA 2.4. *Let T: $K \to K$ be a normal operator on a separable Hilbert space. Suppose $T \cong W_\mu$ via the unitary operator V, where $\mu = (\mu_1, \mu_2, \dots)$. If $Vx = (f_1, f_2, \dots)$, then ν_x is defined by*

$$d\nu_x = \sum_{j=1}^{\infty} |f_j|^2 \, d\mu_j.$$

In particular, if $\mu_1 \gg \mu_2 \gg \dots$, then $\nu_x \ll \mu_1$.

3. Subnormal operators. We now turn to the relationship between a subnormal operator and its minimal normal extension. Suppose S: $H \to H$ is subnormal and n-cyclic with minimal normal extension T: $K \to K$. It is easy to show (see Theorem 3.2) that T is n'-cyclic, where $n' < n$. That n' can be smaller than n is demonstrated dramatically in the following

EXAMPLE 3.1. Let τ be Lebesgue area measure restricted to D, and let $H_n = \mathfrak{M}(1, |z|^2, \dots, |z|^{2n}; W_\tau)$. Note that if $s \neq j$, then $(z^j|z|^{2k}, z^s|z|^{2t})_\tau = 0$ for $0 < k, t < n$. Let $T_n = W_\tau|_{H_n}$. It is easily checked that $\{|z|^{2k}\}_{k=0}^{n}$ spans $\ker T_n^*$. Hence, $(\operatorname{ran} T_n)^-$ has codimension $n + 1$, and it follows that T_n is $(n + 1)$-cyclic. On the other hand, the minimal normal extension is cyclic. If

$$H_\infty = \left(\bigcup_{n=0}^{\infty} H_n \right)^- \quad \text{and} \quad T_\infty = W_\tau|_{H_\infty},$$

then T_∞ is a subnormal operator which is not finitely cyclic ($|z|^{2k} \in \ker T_\infty^*$ for each $k > 0$) even though its minimal normal extension is cyclic. □

In view of Theorem 2.3, the subnormal operator S is unitarily equivalent to the restriction of W_μ to a subspace $\tilde{H}$ of $L^2(\mu)$ for some sequence of measures $\mu = (\mu_1, \dots, \mu_{n'})$. Since S is n-cyclic, there exist n elements $F_1, \dots, F_n$ of $\tilde{H}$ such that $\tilde{H} = \mathfrak{M}(F_1, \dots, F_n; W_\mu)$. In order that this information be useful, we must know something more about $F_1, \dots, F_n$.

THEOREM 3.2. *Suppose S: $H \to H$ is subnormal and n-cyclic with minimal normal extension T: $K \to K$. Then T is n'-cyclic, where $n' \leq n$. Furthermore, there exist a sequence of measures $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{n'})$ and elements $F_1, \ldots, F_n$ of $L^2(\boldsymbol{\mu})$ with $\mu_{k+1} \ll \mu_k$, $1 \leq k < n'$, and*

$$\begin{aligned} F_1 &= (1, 0, \ldots, 0), \\ F_2 &= (\psi_{21}, 1, 0, \ldots, 0), \\ &\vdots \\ F_{n'} &= (\psi_{n'1}, \ldots, \psi_{n',n'-1}, 1), \end{aligned} \tag{3}$$

such that $S \cong W_{\boldsymbol{\mu}}|_{\tilde{H}}$, where $\tilde{H} = \mathfrak{M}(F_1, \ldots, F_n; W_{\boldsymbol{\mu}})$.

PROOF. Choose n vectors $x_1, \ldots, x_n$ in H such that $H = \mathfrak{M}(x_1, \ldots, x_n; S)$. Since T is the minimal normal extension of S, we have $K = \mathfrak{M}(x_1, \ldots, x_n; T)$. By Lemma 2.1, T is n'-cyclic, where $n' \leq n$. Define $\tilde{x}_k$ as in the proof of Lemma 2.1. Of course, one of the vectors $\tilde{x}_k$ may be zero. On the other hand, by reordering the set $\{x_1, \ldots, x_n\}$ if necessary, we may assume that $\tilde{x}_1, \ldots, \tilde{x}_{n''}$ are not zero, while (if $n'' < n$) $\tilde{x}_{n''+1} = \cdots = \tilde{x}_n = 0$. For $1 \leq k \leq n''$, there exist a measure μ_k and a unitary operator V_k: $\mathfrak{M}(\tilde{x}_k; T) \to L^2(\mu_k)$ such that $V_k \tilde{x}_k = 1$ and $T|_{\mathfrak{M}(\tilde{x}_k;T)} \cong W_{\mu_k}$ via V_k. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{n''})$ and define V: $K \to L^2(\boldsymbol{\mu})$ by $V(y_1 \oplus \cdots \oplus y_{n''}) = (V_1 y_1, \ldots, V_{n''} y_{n''})$, so that $T \cong W_{\boldsymbol{\mu}}$ via V. Let $F_k = V x_k$. By construction $F_1, \ldots, F_{n''}$ have the form of (3) with n' replaced by n''. If $\tilde{H} = \mathfrak{M}(F_1, \ldots, F_n; W_{\boldsymbol{\mu}})$, then $S \cong W_{\boldsymbol{\mu}}|_{\tilde{H}}$.

For arbitrary choice of the set $\{x_1, \ldots, x_n\}$ it may happen that $n'' > n'$ (see the following example). We will show, however, that the set $\{x_1, \ldots, x_n\}$ could have been chosen in such a way that $\mu_{k+1} \ll \mu_k$, $1 \leq k \leq n''$. This assumption will imply (Lemma 2.2) that $W_{\boldsymbol{\mu}}$ is n''-cyclic. Since T is n'-cyclic, we will have $n' = n''$, and the proof will be complete.

The basic idea is first to modify x_1 so that with its new value, $\mu_k \ll \mu_1$, $2 \leq k \leq n''$. Next, x_2 is modified so that $\mu_k \ll \mu_2$, $3 \leq k \leq n''$, and so on. We will illustrate this procedure for this case $n'' = 3$. Choose $\alpha_3 \neq 0$ such that $|1 + \alpha_3\psi_{32}| > 0$ a.e. $[\mu_2]$, and then choose $\alpha_2 \neq 0$ such that $|1 + \alpha_2(\psi_{21} + \alpha_3\psi_{31})| > 0$ a.e. $[\mu_1]$. Now, replace x_1 by $x_1 + \alpha_2 x_2 + \alpha_2\alpha_3 x_3$. This has the effect of replacing μ_1 by μ_1' where

$$d\mu_1' = |1 + \alpha_2\psi_{21} + \alpha_2\alpha_3\psi_{31}|^2 d\mu_1 + |\alpha_2|^2 |1 + \alpha_3\psi_{32}|^2 d\mu_2 + |\alpha_2\alpha_3|^2 d\mu_3.$$

This replacement will also effect the definitions of $\tilde{x}_2$ and $\tilde{x}_3$ as well as the measures μ_2 and μ_3 and the functions ψ_{jk}. The important fact is that with the new definitions of μ_1, μ_2 and μ_3 we have $\mu_2 \ll \mu_1$ and $\mu_3 \ll \mu_1$.

If $n' > 1$, we may still assume that $\tilde{x}_2 \neq 0$ by interchanging x_2 and x_3 if necessary. Choose $\beta_3 \neq 0$ such that $|1 + \beta_3\psi_{32}| > 0$ a.e. $[\mu_2]$, and replace x_2

by $x_2 + \beta_3 x_3$. This has the effect of replacing $\tilde{x}_2$ by $V^*(0, 1 + \beta_3\psi_{32}, \beta_3)$, and hence of replacing μ_2 by μ_2', where

$$d\mu_2' = |1 + \beta_3\psi_{32}|^2\, d\mu_2 + |\beta_3|^2\, d\mu_3.$$

Thus we have $\mu_1 \gg \mu_2 \gg \mu_3$. □

The following example shows that we can have $n' < n''$ in the above proof.

EXAMPLE. Let $E_1 = \{z: |z| < \frac{2}{3}\}$, let $E_2 = \{z: \frac{1}{3} < |z| < 1\}$, and set $g_i = \chi_{E_i}$, $i = 1, 2$. Let $H = \mathfrak{M}(g_1, g_2; W_\tau)$ (τ is defined in Example 3.1), and set $S = W_\tau|_H$. Then $g_i \in \ker S^*$, $i = 1, 2$; hence, S is 2-cyclic with cyclic normal extension W_τ. On the other hand, if we let $x_i = g_i$ in the above proof, then we have $n'' = 2$. □

The pair $(\mu; F_1, \ldots, F_n)$ will be called a *standard representation* for the n-cyclic subnormal operator S if

(1) $\mu = (\mu_1, \ldots, \mu_{n'})$ is a sequence of measures with $n' \leq n$ and $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_{n'}$;

(2) $F_1, \ldots, F_n$ are elements of $L^2(\mu)$ with $F_1, \ldots, F_{n'}$ having the form (3); and

(3) S is unitarily equivalent to $W_\mu|_{\tilde{H}}$, where $\tilde{H} = \mathfrak{M}(F_1, \ldots, F_n; W_\mu)$.

In view of the previous theorem, every n-cyclic subnormal operator has a standard representation. One obvious example of an n-cyclic subnormal operator is $S = U_{\mu_1} \oplus U_{\mu_2} \oplus \cdots \oplus U_{\mu_n}$, where μ_j is a measure with compact support, $1 \leq j \leq n$, and $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_n$. Let $\mu = (\mu_1, \ldots, \mu_n)$ and let $F_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the "1" appearing in the jth position). A standard representation for S is $(\mu; F_1, \ldots, F_n)$. Let $H^2(\mu) = \mathfrak{M}(F_1, \ldots, F_n; W_\mu)$ and let $U_\mu = W_\mu|_{H^2(\mu)}$ so that $S \cong U_\mu$. In particular, if $\mu_j = m$, $1 \leq j \leq n$, then we will write $\mathbf{m}^n$ for μ. The operator $U_{\mathbf{m}^n}$ is just the unilateral shift of multiplicity n. We will study the subnormal operators quasisimilar to $U_{\mathbf{m}^n}$ in the next section.

COROLLARY 3.3. *Suppose S: $H \to H$ is subnormal and finitely cyclic with minimal normal extension T: $K \to K$. Then there exists $x \in H$ such that $\nu_y \ll \nu_x$ for each $y \in K$.*

PROOF. Let $(\mu; F_1, \ldots, F_n)$ be a standard representation for S, where $\mu = (\mu_1, \ldots, \mu_{n'})$. By Lemma 2.4, we have $\nu_G \ll \mu_1$ for each $G \in L^2(\mu)$. But $\nu_{F_1} = \mu_1$. □

For a beautiful proof of the following proposition, see Morrel [11].

PROPOSITION 3.4. *If T: $H \to H$, then T has reducing subspaces H_0 and H_1 such that $T|_{H_0}$ is normal, $T|_{H_1}$ is completely nonnormal, and $H = H_0 \oplus H_1$. Furthermore,*

$$H_0 = \bigcap_{j,k>0} \ker(T^jT^{*k} - T^{*k}T^j).$$

Proposition 3.4 yields a canonical decomposition of an operator T into a normal part and a completely nonnormal part. We will call this decomposition the *normal decomposition* of T. We should mention that a completely nonnormal subnormal operator is usually said to be *completely subnormal* or *pure*.

An operator S is said to be dominant (cf. [12]) if $\operatorname{ran}(S - \lambda) \subset \operatorname{ran}(S - \lambda)^*$ for each $\lambda \in \mathbf{C}$. In particular, every hyponormal operator is dominant.

PROPOSITION 3.5. *Let S and T be dominant operators with normal decompositions $S_0 \oplus S_1$ on $H = H_0 \oplus H_1$ and $T_0 \oplus T_1$ on $K = K_0 \oplus K_1$, respectively. Suppose there exist operators X: $H \to K$ and Y: $K \to H$ such that X and Y have no kernel, $XS = TX$, and $SY = YT$. Then $S_0 \cong T_0$.*

PROOF. Let

$$H_2 = (YK_0)^-, \qquad K_2 = (XH_0)^-,$$
$$X_0 = X|_{H_0}: H_0 \to K_2, \quad Y_0 = Y|_{K_0}: K_0 \to H_2,$$
$$S_2 = S|_{H_2}, \qquad T_2 = T|_{K_2}.$$

By [12, Theorem 1] S_2 and T_2 are normal. From [12, Lemma 2] it follows that H_2 and K_2 reduce S_2 and T_2, respectively. Therefore, $H_2 \subset H_0$ and $K_2 \subset K_0$. Since X_0 and Y_0 have no kernel, from [5, Lemma 4.1] it follows that $S_0 \cong T_2$ and $T_0 \cong S_2$. Hence [9] $S_0 \cong T_0$. □

Let T: $K \to K$ be a normal operator on a separable Hilbert space with spectral measure $\mathbf{P}$ so that $T = \int z \, d\mathbf{P}(z)$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$ be a sequence of measures, and let V: $K \to L^2(\boldsymbol{\mu})$ be a unitary operator such that $T \cong W_{\boldsymbol{\mu}}$ via V. Let $L^2_E(\boldsymbol{\mu})$ be those elements F of $L^2(\boldsymbol{\mu})$ for which $F = 0$ a.e. $[\boldsymbol{\mu}]$ on $\mathbf{C} \setminus E$. Then $\mathbf{P}(E)$ (or, more precisely, $V\mathbf{P}(E)V^*$) is the projection of $L^2(\boldsymbol{\mu})$ onto $L^2_E(\boldsymbol{\mu})$. In particular, E is a set of spectral measure zero for T if, and only if, $\mu_j(E) = 0$ for each $j > 1$.

A contraction A operating on H is said to be completely nonunitary (c.n.u.) if no subspace of H reduces A to a unitary operator.

PROPOSITION 3.6. *Suppose S is a c.n.u. subnormal contraction on a separable Hilbert space H with minimal normal extension T: $K \to K$. If $E \subset \mathfrak{T}$ is a set of Lebesgue measure zero, then E is a set of spectral measure zero for T.*

PROOF. First, suppose S is n-cyclic with standard representation $(\boldsymbol{\mu}; F_1, \ldots, F_n)$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{n'})$. Since

$$W_{\boldsymbol{\mu}}|_{\mathfrak{M}(F_1; W_{\boldsymbol{\mu}})} \cong U_{\mu_1},$$

the operator U_{μ_1} is c.n.u. By a result of Clary [4, Lemma 4.5] this is possible only if $\mu_1 \ll m$. Therefore, $\mu_j|_{\mathfrak{T}} \ll m$ because $\mu_j \ll \mu_1$. The proof is complete for the finitely cyclic case.

For the general case, let $\{e_1, e_2, \dots\}$ be an orthonormal basis of H, let $f_1 = e_1$, and let f_k be the orthogonal projection of e_k into $K \ominus \mathfrak{M}(e_1, \dots, e_{k-1}; T)$, so that $K = \mathfrak{M}(f_1; T) \oplus \mathfrak{M}(f_2; T) \oplus \dots$. We must show that $\nu_{f_k}|_{\mathfrak{T}} \ll m$. Let $H_n = \mathfrak{M}(e_1, \dots, e_n; S)$, and let $S_n = S|_{H_n}$. The minimal normal extension of S_n is $T|_{K_n}$, where

$$K_n = \mathfrak{M}(e_1, \dots, e_n; T) = \mathfrak{M}(f_1; T) \oplus \cdots \oplus \mathfrak{M}(f_n; T).$$

By the first part of this proof, $\nu_{f_k}|_{\mathfrak{T}} \ll m$, $1 \leq k \leq n$. Since n was arbitrary, the proof is complete. □

4. Subnormal operators of type $\mathcal{S}_n$. We now come to the main results. Suppose S and S' are quasisimilar subnormal operators. In the previous section we showed that their normal parts must be unitarily equivalent. In this section we find additional hypotheses which imply that their completely subnormal parts must be quasisimilar. A complete characterization of those subnormal operators quasisimilar to a unilateral shift of finite multiplicity is also given in Theorem 4.4.

PROPOSITION 4.1. *Let S: $H \to H$ be a subnormal operator with standard representation (μ; $F_1, \dots, F_n$), where $\mu = (\mu_1, \dots, \mu_{n'})$. Suppose* supp $\mu_j \subseteq \overline{D}$ *and $\mu_j|_{\mathfrak{T}} \ll m$, $1 \leq j \leq n'$. In addition, suppose there is a bounded operator X: $H \to H^2(\mathbf{m}^n)$ with dense range satisfying $XS = U_{\mathbf{m}^n}X$. Then*

(a) $n' = n$,

(b) *X is one-to-one,*

(c) *$\mu_{j|\mathfrak{T}}$ and m are mutually absolutely continuous.*

PROOF. Let $\mu_0 = (\mu_{10}, \dots, \mu_{n'0})$, where $\mu_{j0} = \mu_j|_{\mathfrak{T}}$. Let $H_0 = \mathfrak{M}(F_1, \dots, F_n; W_{\mu_0})$, and set $S_0 = W_{\mu_0}|_{H_0}$. As in [4, Lemma 4.5γ], the operator X induces an operator X_0: $H_0 \to H^2(\mathbf{m}^n)$ with $X_0F = XF$ for each $F \in \mathfrak{M}(F_1, \dots, F_n; W_\mu)$. Now X_0 has an extension $\tilde{X}_0$: $L^2(\mu_0) \to L^2(\mathbf{m}^n)$ such that $\tilde{X}_0|_{H_0} = X_0$ and $\tilde{X}_0 W_{\mu_0} = W_{\mathbf{m}^n}\tilde{X}_0$ (cf. [5, Corollary 5.1]). By a lemma of Douglas [5, Lemma 4.1], $(\ker \tilde{X}_0)^\perp$ reduces W_{μ_0}, $(\operatorname{ran} \tilde{X}_0)^-$ reduces $W_{\mathbf{m}^n}$, and

$$W_{\mu_0}|_{(\ker \tilde{X}_0)^\perp} \cong W_{\mathbf{m}^n}|_{(\operatorname{ran} \tilde{X}_0)^-}.$$

But $(\operatorname{ran} \tilde{X}_0)^- \supset H^2(\mathbf{m}^n)$, and so $(\operatorname{ran} \tilde{X}_0)^- = L^2(\mathbf{m}^n)$.

Suppose $F \in \ker \tilde{X}_0$, $F \neq 0$. Let $K = \mathfrak{M}(F; W_{\mu_0})$. Then

$$W_{\mu_0}|_{(\ker \tilde{X}_0)^\perp} \oplus W_{\mu_0}|_K \cong W_{\mathbf{m}^n} \oplus W_{\mu_0}|_K.$$

But $\nu_F \ll m$; therefore $W_{\mathbf{m}^n} \oplus W_{\mu_0}|_K$ is $(n+1)$-cyclic by Lemma 2.2. This contradiction shows that $\tilde{X}_0$ and, a fortiori, X have no kernel. Thus, $W_{\mu_0} \cong W_{\mathbf{m}^n}$ and $n' = n$. It now follows that $\mu_{j|\mathfrak{T}}$ and m are mutually absolutely continuous. □

Suppose an operator S is subnormal and n-cyclic with standard representation $(\mu; F_1, \ldots, F_n)$, where $\mu = (\mu_1, \ldots, \mu_{n'})$. The operator S is said to be of type $\mathcal{S}_n$ if and only if $n' = n$ and μ_j is of type $\mathcal{S}$, $1 < j < n$. The next two lemmas together show that if S is of type $\mathcal{S}_n$, then S is quasisimilar to the unilateral shift of multiplicity n.

LEMMA 4.2. *Suppose S: $H \to H$ is a subnormal operator of type $\mathcal{S}_n$. Then there is a quasiinvertible operator X: $H \to H^2(\mathbf{m}^n)$ such that $XS = U_{\mathbf{m}^n}X$.*

PROOF. Let $(\mu; F_1, \ldots, F_n)$ be a standard representation for S, where

$$F_k = (\psi_{k1}, \ldots, \psi_{k,k-1}, 1, 0, \ldots, 0).$$

Let $\tilde{H} = \mathfrak{M}(F_1, \ldots, F_n; W_\mu)$ and set $\tilde{S} = W_\mu|_{\tilde{H}}$. For convenience, let $\psi_{kk} = 1$ and $\psi_{kj} = 0$ for $j > k$, $1 < k < n$. We assert that there exist outer functions $g_1, \ldots, g_n$ in H^2 such that for any set of n polynomials $\{p_1, \ldots, p_n\}$ we have

$$|p_i g_i|^2 < \sum_{j=1}^{n} \left| \sum_{k=j}^{n} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} \quad \text{a.e. } [m], \quad 1 < i < n.$$

Suppose for now that this assertion is true. Define $G_1, \ldots, G_n \in H^2(\mathbf{m}^n)$ by $G_k = (0, \ldots, 0, g_k, 0, \ldots, 0)$, the g_k appearing in the kth position. Then we may define X: $\tilde{H} \to H^2(\mathbf{m}^n)$ by $XF_k = G_k$ and $X\tilde{S} = U_{\mathbf{m}^n}X$. A straightforward computation shows that this defines a bounded operator. Since $g_1, \ldots, g_n$ are outer, X has dense range. Suppose $(f_1, \ldots, f_n) \in \tilde{H}$ and $X(f_1, \ldots, f_n) = 0$. Choose polynomials p_{jk}, $1 < k < n$, $j > 1$, such that $\sum_{k=1}^n p_{jk}F_k \to (f_1, \ldots, f_n)$ in $\tilde{H}$ and such that $X(p_{j1}F_1 + \cdots + p_{jn}F_n) \to 0$ pointwise a.e. $[\mathbf{m}^n]$. This means that $p_{jk}g_k \to 0$ pointwise a.e. $[m]$ on $\mathfrak{T}$ and $p_{jk}g_k \to 0$ pointwise everywhere in D. Since g_k is outer, this is possible only if $\sum_{k=1}^n p_{jk}F_k \to 0$ pointwise a.e. $[\mu]$; that is, $(f_1, \ldots, f_n) = 0$.

We now prove the assertion above by induction on n. If $n = 1$, let g_1 be an outer function with $|g_1|^2 = d\mu_1/dm$. Suppose $n > 2$ and suppose the result is true with $n - 1$ in place of n. Then there are outer functions $\tilde{g}_1, \ldots, \tilde{g}_{n-1}$ in H^2 such that

$$|p_i \tilde{g}_i|^2 < \sum_{j=1}^{n-1} \left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} \quad \text{a.e. } [m]$$

for any set of $(n - 1)$ polynomials $p_1, \ldots, p_{n-1}$. Let g_n be an outer function such that $|g_n|^2 = d\mu_n/dm$. For $1 < i < n - 1$, let

$$\varphi_i = |\tilde{g}_i|^2 \min\left\{ \frac{1}{2n-1}, \left[2(n-1) \sum_{j=1}^{n-1} |\psi_{nj}|^2 \frac{d\mu_j}{dm} \right]^{-1} \frac{d\mu_n}{dm} \right\}.$$

Then $\int \log \varphi_i \, dm > -\infty$. Indeed, fix i, $1 < i < n - 1$, and let

$$E = \left\{ z \in \mathfrak{T} : \varphi_i(z) = \frac{1}{2n-1} |\tilde{g}_i(z)|^2 \right\}.$$

Then

$$\int_E \log \varphi_i \, dm = \int_E \log \frac{|\tilde{g}_i|^2}{2n-1} \, dm > -\infty$$

while

$$\int_{\mathfrak{T}\setminus E} \log \varphi_i \, dm = -\int_{\mathfrak{T}\setminus E} \log\left(2(n-1)\sum_{j=1}^{n-1} |\psi_{nj}|^2 \frac{d\mu_j}{dm}\right) dm + \int_{\mathfrak{T}\setminus E} \log \frac{d\mu_n}{dm} \, dm$$

$$> -\int_{\mathfrak{T}\setminus E}\left(2(n-1)\sum_{j=1}^{n-1} |\psi_{nj}|^2 \frac{d\mu_j}{dm}\right) dm + \int_{\mathfrak{T}\setminus E} \log \frac{d\mu_n}{dm} \, dm > -\infty.$$

Therefore, there is an outer function $g_i \in H^2$ such that $|g_i|^2 = \varphi_i$.

Now let $\{p_1, \ldots, p_n\}$ be a set of n polynomials and suppose $1 \leqslant i \leqslant n-1$. By our choice of g_i, we have

$$|p_i g_i|^2 \leqslant (2n-1)^{-1} |p_i \tilde{g}_i|^2 \leqslant (2n-1)^{-1} \left\{ \sum_{j=1}^{n-1} \left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} \right\} \quad \text{a.e. } [m]. \tag{4}$$

Suppose

$$(2n-1)^{-1} \left\{ \sum_{j=1}^{n-1} \left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} \right\} > \sum_{j=1}^{n-1} \left| \sum_{k=j}^{n} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} \tag{5}$$

on a set E_0 of positive Lebesgue measure. We will show that on this set

$$|p_i g_i|^2 \leqslant |p_n|^2 \frac{d\mu_n}{dm} \quad \text{a.e. } [m].$$

(The following inequalities will all hold a.e. [m] on E_0.) Indeed, by (5) for $1 \leqslant j \leqslant n-1$

$$\left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} < \frac{2}{2n-1} \sum_{i=1}^{n-1} \left| \sum_{k=i}^{n-1} p_k \psi_{ki} \right|^2 \frac{d\mu_i}{dm} + 2|p_n \psi_{nj}|^2 \frac{d\mu_j}{dm}.$$

(We are using here the following fact: if $|z+w|^2 x < y$, then $|z|^2 x < 2y + 2|w|^2 x$, where z and w are complex and x and y are positive reals.) Addition of these inequalities yields

$$\sum_{j=1}^{n-1} \left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} < \frac{2(n-1)}{2n-1} \left[\sum_{j=1}^{n-1} \left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} + \sum_{j=1}^{n-1} |p_n \psi_{nj}|^2 \frac{d\mu_j}{dm} \right].$$

Hence,

$$\sum_{j=1}^{n-1} \left| \sum_{k=j}^{n-1} p_k \psi_{kj} \right|^2 \frac{d\mu_j}{dm} < |p_n|^2 \left(2(n-1) \sum_{j=1}^{n-1} |\psi_{nj}|^2 \frac{d\mu_j}{dm} \right).$$

Combining this inequality with (4) yields

$$|p_i\tilde{g}_i|^2 < |p_n|^2\left(2(n-1)\sum_{j=1}^{n-1}|\psi_{nj}|^2\frac{d\mu_j}{dm}\right)$$

or

$$|p_i|^2|\tilde{g}_i|^2\left(2(n-1)\sum_{j=1}^{n-1}|\psi_{nj}|^2\frac{d\mu_j}{dm}\right)^{-1}\frac{d\mu_n}{dm} < |p_n|^2\frac{d\mu_n}{dm}.$$

Finally, from the definition of g_i,

$$|p_i|^2|g_i|^2 < |p_n|^2\frac{d\mu_n}{dm} \quad \text{a.e. } [m] \quad \text{on } E_0. \quad \square$$

LEMMA 4.3. *Let S: $H \to H$ be subnormal with standard representation (μ; $F_1, \ldots, F_n$), where $\mu = (\mu_1, \ldots, \mu_n)$. Suppose that, for $1 < j < n$,*

(a) supp $\mu_j \subseteq \bar{D}$,

(b) $\mu_j|_{\mathfrak{T}} \ll m$,

(c) μ_j *is not carried by the zero set of any element of* H^2.

Then there exists a quasiinvertible operator Y: $H^2(\mathbf{m}^n) \to H$ such that $YU_{\mathbf{m}^n} = SY$.

PROOF. Let $\tilde{H} = \mathfrak{M}(F_1, \ldots, F_n;\ W_\mu)$ and set $\tilde{S} = W_\mu|_{\tilde{H}}$. Let $F_k = (\psi_{k1}, \ldots, \psi_{kn})$, $1 < k < n$. For convenience, we have set $\psi_{kk} = 1$, $1 < k < n$ and $\psi_{jk} = 0$, $1 < j < k < n$. Define a measure ν_k by

$$d\nu_k = \sum_{j=1}^{n}|\psi_{kj}|^2\, d\mu_j.$$

For each k, $1 < k < n$, there exists a function $R_k \in L^1(m)$ such that $R_k > 0$ a.e. $[m]$ and

$$\int_D |f|^2\, d\nu_k < \int_{\mathfrak{T}} |f|^2 R_k\, dm < \infty$$

for every $f \in H^\infty$. (Cf. [4, Lemma 4.4α] or [6, pp. 454–455]. The integral on the left-hand side is over the open disk only.) Let g_k be an outer function in H^∞ such that

$$|g_k|^2 = \left(1 + R_k + \frac{d\nu_k}{dm}\right)^{-1}, \quad 1 < k < n.$$

Define Y: $H^2(\mathbf{m}^n) \to \tilde{H}$ by

$$Y(0, \ldots, 0, 1, 0, \ldots, 0) = g_k F_k \quad \text{and} \quad YU_{\mathbf{m}^n} = \tilde{S}Y.$$

For arbitrary polynomials $p_1, \ldots, p_n$,

$$\begin{aligned}\|Y(p_1, \ldots, p_n)\|_{\mu}^2 &= \left\| \sum_{k=1}^{n} p_k g_k F_k \right\|_{\mu}^2 \\ &= \sum_{j=1}^{n} \int \left| \sum_{k=1}^{n} p_k g_k \psi_{kj} \right|^2 d\mu_j \\ &\leq n \sum_{k=1}^{n} \int |p_k|^2 |g_k|^2 \left(\sum_{j=1}^{n} |\psi_{kj}|^2 \, d\mu_j \right) \\ &\leq n \sum_{k=1}^{n} \int |p_k|^2 |g_k|^2 \left(R_k + \frac{d\nu_k}{dm} \right) dm \\ &\leq n \|(p_1, \ldots, p_n)\|_{\mathbf{m}^n}^2.\end{aligned}$$

Thus Y is bounded. By our choice of $g_1, \ldots, g_n$, their reciprocals belong to H^2; hence, Y has dense range. Suppose

$$0 = Y(h_1, \ldots, h_n) = \left(\sum_{k=1}^{n} h_k g_k \psi_{k1}, \ldots, \sum_{k=n-1}^{n} h_k g_k \psi_{k,n-1}, h_n g_n \right).$$

Since $|g_n| > 0$ a.e. $[\mu_n]$, the function h_n must vanish a.e. $[\mu_n]$. By hypothesis (c), we must have $h_n = 0$ in H^2. But then the $(n-1)$st coordinate of $Y(h_1, \ldots, h_n)$ is $h_{n-1} g_{n-1}$. By the same argument $h_{n-1} = 0$. Continuing in this way, we see that $h_1 = \cdots = h_n = 0$; that is, Y is one-to-one. □

THEOREM 4.4. *Suppose S is subnormal. Then $S \sim\sim U_{\mathbf{m}^n}$ if and only if S is of type $\mathbb{S}_n$.*

PROOF. If S is of type $\mathbb{S}_n$, then Lemma 4.2 and Lemma 4.3 together imply that $S \sim\sim U_{\mathbf{m}^n}$. Conversely, suppose $S \sim\sim U_{\mathbf{m}^n}$. Trivially, S is n-cyclic, and by Proposition 3.5, S is c.n.u. Let $(\mu; F_1, \ldots, F_n)$ be a standard representation for S, where $\mu = (\mu_1, \ldots, \mu_{n'})$. Let $H = \mathfrak{M}(F_1, \ldots, F_n; W_\mu)$. Without loss of generality, we may assume that $S = W_\mu|_H$. There exist quasiinvertible operators X: $H \to H^2(\mathbf{m}^n)$ and Y: $H^2(\mathbf{m}^n) \to H$ such that $S \sim\sim U_{\mathbf{m}^n}$ via (X, Y). The proof will be by induction on n. For $n = 1$, the result is Theorem 4.5 of [4]. Suppose $n > 1$ and suppose the theorem is true with $n - 1$ in place of n.

Since $\sigma(S) = \sigma(U_{\mathbf{m}^n}) = \overline{D}$, it is easy to see that supp $\mu_k \subset \overline{D}$, $1 \leq k \leq n'$. Let $\mu_{k0} = \mu_k|_{\mathbb{T}}$ and set $\mu_0 = (\mu_{10}, \ldots, \mu_{n'0})$. By Proposition 3.6, $\mu_{k0} \ll m$, $1 \leq k \leq n'$. Thus, we may apply Proposition 4.1 to conclude that $n' = n$ and that $m \ll \mu_{k0}$, $1 \leq k \leq n'$. Let $H_0 = \mathfrak{M}(F_1, \ldots, F_n; W_{\mu_0})$, and let $S_0 = W_{\mu_0}|_{H_0}$. As in the proof of Proposition 4.1, the operator X induces an operator X_0: $H_0 \to H^2(\mathbf{m}^n)$ such that $X_0 F = XF$ for each $F \in H$ and $X_0 S_0 = U_{\mathbf{m}^n} X_0$. In view of Proposition 4.1 and Lemma 4.3, X_0 has no kernel and $S_0 \sim\sim U_{\mathbf{m}^n}$.

To apply the inductive hypothesis, let $H_1 = \mathfrak{M}(F_1, \ldots, F_{n-1}; S_0)$, let $S_1 = S_0|_{H_1}$, and let $X_1 = X_0|_{H_1}$. Then $U_{\mathbf{m}^n}|_{(\operatorname{ran} X_1)^-}$ is an $(n-1)$-cyclic isometry with no unitary part; that is, $U_{\mathbf{m}^n}|_{(\operatorname{ran} X_1)^-} \cong U_{\mathbf{m}^{n-1}}$.

Again by Proposition 4.1 and Lemma 4.3, $S_1 \sim\sim U_{\mathbf{m}^{n-1}}$. A standard representation for S_1 is $(\tilde{\mu}_0; F_1, \ldots, F_{n-1})$, where $\tilde{\mu}_0 = (\mu_{10}, \ldots, \mu_{n-1,0})$. Hence by the inductive hypothesis, the measures $\mu_1, \ldots, \mu_{n-1}$ are of type $\mathcal{S}$. Since quasisimilar isometries are unitarily equivalent [8], there exists a unitary operator V_1: $H_1 \to H^2(\mathbf{m}^{n-1})$ such that $S_1 \cong U_{\mathbf{m}^{n-1}}$ via V_1. The operator V_1 extends to a unitary operator $\tilde{V}_1$: $L^2(\tilde{\mu}_0) \to L^2(\mathbf{m}^{n-1})$ such that $W_{\tilde{\mu}_0} \cong W_{\mathbf{m}^{n-1}}$ via $\tilde{V}_1$ [5, Corollary 5.1]. Let $\varphi_n = (d\mu_n/dm)^{1/2}$ and define a unitary operator V_2: $L^2(\mu_0) \to L^2(\mathbf{m}^n)$ by $V_2(f_1, \ldots, f_n) = (\tilde{f}_1, \ldots, \tilde{f}_{n-1}, \varphi_n f_n)$, where $(\tilde{f}_1, \ldots, \tilde{f}_{n-1}) = \tilde{V}_1(f_1, \ldots, f_{n-1})$. Finally, let $H_2 = V_2 H_0$, let $S_2 = W_{\mathbf{m}^n}|_{H_2}$, and set $\tilde{F}_1 = (1, 0, \ldots, 0)$, $\tilde{F}_2 = (0, 1, 0, \ldots, 0), \ldots, \tilde{F}_{n-1} = (0, \ldots, 0, 1, 0)$ and $\tilde{F}_n = V_2 F_n = (\varphi_1, \ldots, \varphi_{n-1}, \varphi_n)$.

Then $H_2 = \mathfrak{M}(\tilde{F}_1, \ldots, \tilde{F}_n; W_{\mathbf{m}^n})$, and again using the fact that quasisimilar isometries are unitarily equivalent, there is a unitary operator V_3: $H_2 \to H^2(\mathbf{m}^n)$ such that $S_2 \cong U_{\mathbf{m}^n}$ via V_3. Let $(g_{j1}, \ldots, g_{jn}) = V_3 \tilde{F}_j$, $1 < j < n$. By the unitary equivalence the following equalities hold a.e. $[m]$:

$$\sum_{k=1}^{n} g_{ik}\bar{g}_{jk} = 0, \qquad 1 < i < j < n,$$

$$\sum_{k=1}^{n} |g_{jk}|^2 = 1, \qquad 1 < j < n,$$

$$\sum_{k=1}^{n} \bar{g}_{jk} g_{nk} = \varphi_j, \qquad 1 < j < n,$$

and

$$\sum_{k=1}^{n} |g_{nk}|^2 = \sum_{k=1}^{n} |\varphi_k|^2.$$

For example, if $1 < i < j < n$ and p and q are arbitrary polynomials, then

$$\begin{aligned} 0 = (p\tilde{F}_i, q\tilde{F}_j)_{\mu_0} &= (pV_3\tilde{F}_i, qV_3\tilde{F}_j)_{\mathbf{m}^n} \\ &= \sum_{k=1}^{n} \int (pg_{ik})(\overline{qg_{jk}})\, dm = \int p\bar{q}\left(\sum_{k=1}^{n} g_{ik}\bar{g}_{jk}\right) dm. \end{aligned}$$

In particular, the Fourier coefficients of the function $\sum_{k=1}^{n} g_{ik}\bar{g}_{jk}$ are all zero, which is possible only if $\sum_{k=1}^{n} g_{ik}\bar{g}_{jk} = 0$ a.e. $[m]$.

To finish the proof, we must show that $\int \log (d\mu_n/dm) > -\infty$. We assert that

$$\frac{d\mu_n}{dm} = |\varphi_n|^2 = |\det(g_{jk})|^2 \quad \text{a.e. } [m].$$

Indeed,

$$|\det(g_{jk})|^2 = \det(g_{jk})\overline{\det(g_{jk})} = \det(g_{jk})\det(g_{jk})^* = \det\left(\sum_{k=1}^{n} g_{ik}\overline{g_{jk}}\right)$$

$$= \det\begin{bmatrix} 1 & & & & \bar{\varphi}_1 \\ & 1 & & & \bar{\varphi}_2 \\ & & \ddots & & \vdots \\ & & & 1 & \bar{\varphi}_{n-1} \\ \varphi_1 & \varphi_2 & \cdots & \varphi_{n-1} & \sum_{k=1}^{n}|\varphi_k|^2 \end{bmatrix}$$

$$= |\varphi_n|^2 = \frac{d\mu_n}{dm}.$$

But $\det(g_{jk}) \in H^{2/n}$ and since $d\mu_n/dm$ is not the zero function,

$$\int \log \frac{d\mu_n}{dm}\, dm > -\infty. \quad \square$$

The following example shows that Theorem 4.4 does not generalize in any obvious way to include the case in which S is not finitely cyclic.

EXAMPLE. Let $\mu = (m, m, \dots)$. Let $\varphi \in L^2(m)$ be a cyclic vector for W_m, let φ_1 be the orthogonal projection of φ into H^2, and let $\varphi_2 = \varphi - \varphi_1$. Define $F_j \in L^2(\mu)$ as follows:

$$F_1 = (1, 0, \dots),$$
$$F_2 = (\varphi_2, 1, 0, \dots),$$
$$F_3 = (0, \bar{z}, 1, 0, \dots),$$
$$F_4 = (0, 0, \bar{z}, 1, 0, \dots),$$
$$\vdots$$

Let $H = \mathfrak{M}(\{F_j\}_{j=1}^{\infty}; W_\mu)$ and set $S = W_\mu|_H$. The operator S is an isometry, and in view of Theorem 4.4, one might suspect that S is quasisimilar to a unilateral shift of infinite multiplicity. But, this is not the case because $F = (\varphi, 0, 0, \dots) \in H$. Indeed, $(\varphi_1, 0, \dots)$ is obviously in H. Also,

$$F_2 - zF_3 + z^2F_4 - z^3F_5 + \cdots + z^{2n}F_{2n+2}$$
$$= (\varphi_2, 0, \dots, 0, z^{2n}, 0, \dots) \in H.$$

Therefore, $(\varphi_2, 0, \dots) \in H$, and hence $F = (\varphi_1 + \varphi_2, 0, \dots) \in H$. Now $S|_{\mathfrak{M}(F;S)}$ is unitary, and therefore S cannot be quasisimilar to a unilateral shift. $\square$

We now come to the second main result of this section which, under appropriate conditions, extends Proposition 3.5.

THEOREM 4.5. *Let S and S' be subnormal with normal decompositions*

$$S_0 \oplus S_1: H_0 \oplus H_1 \to H_0 \oplus H_1 \quad \text{and} \quad S_0' \oplus S_1': H_0' \oplus H_1' \to H_0' \oplus H_1'.$$

Suppose S_1' is of type $\mathfrak{S}_n$. Then $S \sim\sim S'$ if, and only if, $S_0 \cong S_0'$ and $S_1 \sim\sim S_1'$.

PROOF. If $S_0 \cong S_0'$ and $S_1 \sim\sim S_1'$, then trivially $S \sim\sim S'$. Conversely, suppose $S \sim\sim S'$ via (X, Y). By Proposition 3.5, $S_0 \cong S_0'$. Furthermore, by Theorem 4.4 we may assume that $S_1' \cong U_{m^n}$. Let $X_1 = X|_{H_1}$ and let P be the orthogonal projection of H' onto H_1'. As shown in the proof of Proposition 3.5, $XH_0 \subset H_0'$. It follows that $(PX_1)(H_1)$ is dense in H_1' and that $(PX_1)S_1 = S_1'(PX_1)$. Therefore, S_1 is cyclic of order at least n. On the other hand, repeating this argument with Y in place of X shows that S_1 is cyclic of order at most n. Thus S_1 is n-cyclic. By Proposition 4.1 the operator PX_1 has no kernel and the minimal normal extension of S_1 is n-cyclic. In view of Lemma 4.3, $S_1 \sim\sim S_1'$. Therefore the operator S_1 is of type $\mathfrak{S}_n$ by Theorem 4.4. □

COROLLARY. *Suppose S is subnormal with normal decompositon $S_0 \oplus S_1$. In addition, suppose S_1 is n-cyclic. Then S is quasisimilar to an isometry if, and only if, S_0 is unitary and S_1 is of type $\mathfrak{S}_n$.*

PROOF. If S_0 is unitary and S_1 is of type $\mathfrak{S}_n$, then $S \sim\sim S_0 \oplus U_{m^n}$. Conversely, suppose $S \sim\sim V$, where V is an isometry with Wold decomposition $V_0 \oplus V_1$. Then $S_0 \cong V_0$ (Proposition 3.5); that is, S_0 is unitary. Furthermore, arguing as in the proof of Theorem 4.5, V_1 is a shift of finite multiplicity n. Therefore $S_1 \sim\sim V_1$ and by Theorem 4.4 S_1 is of type $\mathfrak{S}_n$. □

Theorem 4.5 is false when S_1' is a shift of infinite multiplicity even if S_0' is unitary. The author is indebted to Professor Thomas Kriete for pointing out the existence of a measure with the necessary properties for this example.

EXAMPLE. Let μ be a measure such that

(1) supp $\mu \subset \overline{D}$,

(2) $\mu(\mathfrak{T}) > 0$,

(3) $\int \log(d\mu/dm)\, dm = -\infty$,

(4) U_μ is completely subnormal.

(Such measures exist. Indeed, let C_0 be the Cesàro operator and choose μ such that $(I - C_0) \cong U_\mu$. See Theorem 8 and the proof of Theorem 2 in [10].)

Let $\mu_0 = \mu|_{\mathfrak{T}}$ and let $\boldsymbol{\mu}_0 = (\mu_0, \mu_0, \dots)$. Let $\boldsymbol{\nu} = (\mu, m, m, \dots)$ and let $\boldsymbol{\nu}' = (m, m, \dots)$. Then $W_{\boldsymbol{\mu}_0} \oplus U_{\boldsymbol{\nu}} \sim\sim W_{\boldsymbol{\mu}_0} \oplus U_{\boldsymbol{\nu}'}$, but $U_{\boldsymbol{\nu}}$ and $U_{\boldsymbol{\nu}'}$ are not quasisimilar.

PROOF. Define X: $L^2(\boldsymbol{\mu}_0) \oplus H^2(\boldsymbol{\nu}) \to L^2(\boldsymbol{\mu}_0) \oplus H^2(\boldsymbol{\nu}')$ and Y: $L^2(\boldsymbol{\mu}_0) \oplus H^2(\boldsymbol{\nu}') \to L^2(\boldsymbol{\mu}_0) \oplus H^2(\boldsymbol{\nu})$ by

$$X((f_1, f_2, \dots) \oplus (g_1, g_2, \dots)) = (g_1, f_1, f_2, \dots) \oplus (g_2, g_3, \dots)$$

and

$$Y((f_1, f_2, \ldots) \oplus (g_1, g_2, \ldots)) = (f_1, f_2, \ldots) \oplus (\varphi g_1, g_2, \ldots).$$

Here, φ is an outer function in H^∞ such that the mapping $g \mapsto g\varphi$ defines a bounded, quasiinvertible operator from $H^2(m)$ into $H^2(\mu)$ (cf. Lemma 4.3). An easy verification shows that

$$W_{\mu_0} \oplus U_\nu \sim\sim W_{\mu_0} \oplus U_{\nu'} \quad \text{via } (X, Y).$$

Now, suppose there exists a bounded operator X_1: $H^2(\nu) \to H^2(\nu')$ such that $X_1 U_\nu = U_{\nu'} X_1$. Let H be those elements in $H^2(\nu)$ of the form

$$(f, 0, 0, \ldots), \qquad f \in H^2(\mu).$$

By [**10**, Theorem 1] $X_1|_H$ is the zero operator because of condition (3) on μ. Therefore U_ν and $U_{\nu'}$ cannot be quasisimilar.

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